

Comments on “Extremal Cayley digraphs of finite Abelian groups” [Intercon. Networks 12 (2011), no. 1-2, 125–135]

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Abstract

We comment on the paper “Extremal Cayley digraphs of finite Abelian groups” [Intercon. Networks 12 (2011), no. 1-2, 125–135]. In particular, we give some counterexamples to the results presented there, and provide a correct result for degree two.

1 Introduction

For the description of the problem, its applications, used notation, and the theoretical background, see, e.g. [8, 2, 4, 7].

For some given positive numbers, d (diameter) and k (degree), the authors of [5] consider the following numbers:

- Let $m_*(d, k)$ the largest positive integer m (number of vertices) such that there exists an m -element finite Abelian group Γ and a k -element generating subset $A \subset \Gamma$ such that $\text{diam}(\text{Cay}(\Gamma, A)) \leq d$.
- Let $m(d, k)$ the largest positive integer m such that there exists a cyclic group \mathbb{Z}_m and a k -element generating subset $A \subset \mathbb{Z}_m$ such that $\text{diam}(\text{Cay}(\mathbb{Z}_m, A)) \leq d$.

Such authors claim that, for any integer $d \geq 2$, Jia and Hsu [4] proved that

$$m(d, 2) = \left\lfloor \frac{d(d+4)}{3} \right\rfloor + 1, \quad (1)$$

but this was proved around ten years before by the author et al. [6, 7]. Thus, in [6], the following value can be found:

$$m(d, 2) = \left\lceil \frac{(d+2)^2}{3} \right\rceil - 1, \quad (2)$$

which is readily seen to be equivalent to (1). More generally, in Table I of [7] some other optimal values are shown (minimizing the diameter for some fixed number of vertices). A part of such a table is shown next with the corresponding generating sets $\{a, b\}$ of the cyclic groups. (The values in boldface correspond to the ones given by (1) or (2).

$m(d, 2)$	d	a	$b(\text{mod } m)$
$3x^2$	$3x - 1$	1	$3x - 1$
$3x^2 + x$	$3x - 1$	1	$3x$
$3x^2 + 2x$	$3x - 1$	1	$-3x$
$3x^2 + 2x + 1$	$3x$	1	$3x + 1$
$3x^2 + 3x + 1$	$3x$	1	$3x + 2$
$3x^2 + 4x + 1$	$3x$	1	$-3x - 2$
$3x^2 + 4x + 2$	$3x + 1$	1	$3x + 3$
$3x^2 + 5x + 2$	$3x + 1$	1	$3x + 4$
$3x^2 + 6x + 2$	$3x + 1$	1	$-3x + 4$
$(= 3(x+1)^2 - 1)$			

Also, as a main result, Mask, Schneider, and Jia [5, Th. 1.1] claimed that, for any d and k ,

$$m_*(d, k) = m(d, k). \quad (3)$$

However, as shown by the counterexamples in the following section, such a result cannot be true even for degree $k = 2$. This is due to an error in the proof of such a theorem. Namely, the first r equalities in [5, Th. 1.1] should be understood modulo m_j :

$$x_j = \sum_{i=1}^k c_i a_{ij} \pmod{m_j} \quad \text{for } j = 1, 2, \dots, r.$$

Thus, without this condition, the following equality in [5], which should be modulo $m'_{r-1} = m_{r-1}m_r$, does not necessarily holds.

2 Some counterexamples and a result

In [7] it was shown that for degree $k = |A| = 2$, the minimum diameter d of an Abelian group Γ with m vertices is $d_{\min} = \lceil \sqrt{3m} \rceil - 2$ (see [7, Eq. (9)]). That is,

$$m_*(d, 2) \leq \left\lceil \frac{(d+2)^2}{3} \right\rceil. \quad (4)$$

In fact the upper bound is attained when $\Gamma = \mathbb{Z}_{3x} \times \mathbb{Z}_x$, with $x \geq 1$, and $A = \{(1, 0), (-1, 1)\}$, leading to a (2-regular) Cayley digraph on $m = 3x^2$ vertices and diameter $d = 3x - 2$. However, it can be shown that, when $x > 1$, $\text{rank } \Gamma = 2$, so that Γ is not cyclic. In this case, the best result is obtained with the cyclic group \mathbb{Z}_m with $m = \frac{1}{3}(d + 2)^2 - 1$ and generating set $A = \{a, b\}$, as shown in the following table.

k	x	$d = 3x - 2$	$m_*(d, 2) = 3x^2$	$A \subset \mathbb{Z}_{3x} \times \mathbb{Z}_x$	$m(d, 2) = 3x^2 - 1$	$A \subset \mathbb{Z}_m$
2	2	4	12	$\{(1, 0), (-1, 1)\}$	11	$\{1, 3\}$
2	3	7	27	$\{(1, 0), (-1, 1)\}$	26	$\{1, 8\}$
2	4	10	48	$\{(1, 0), (-1, 1)\}$	47	$\{1, 11\}$
2	5	13	75	$\{(1, 0), (-1, 1)\}$	74	$\{1, 14\}$
2	6	16	108	$\{(1, 0), (-1, 1)\}$	107	$\{1, 17\}$

For other values of $m(d, 2)$, see [7, Table II] or the results in [3, 1]. In fact, from the results of these papers, and comparing the values of $m(d, 2)$ in (2) with the upper bound for $m_*(d, 2)$ in (4), one gets the following result for the case of degree $k = 2$:

Proposition 2.1 *For any diameter $d \geq 2$,*

$$m_*(d, 2) = \begin{cases} m(d, 2) + 1, & \text{if } d \equiv 1 \pmod{3}, \\ m(d, 2), & \text{otherwise.} \end{cases} \quad (5)$$

In the case of the above digraph $\text{Cay}(\mathbb{Z}_{3x} \times \mathbb{Z}_x, \{(1, 0), (-1, 1)\})$, it can be shown that the two unique vertices at maximum distance $d = 3x - 2$ from the origin are $(2x, x - 1)$ and $(x, x - 1)$.

Similar counterexamples can be given to prove that the extremal Cayley digraphs with respect to their average distance are not necessarily attained for cyclic groups ([5, Th. 3.1]).

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References

- [1] F. Aguiló, M.A. Fiol, An efficient algorithm to find optimal double loop networks, *Discrete Math.* **138** (1995) 15–29.

- [2] J.-C. Bermond, F. Comellas, and D.F. Hsu, Distributed loop computer networks: a survey, *J. Parallel Distribut. Comput.* **24** (1995) 2–10.
- [3] P. Esqué, F. Aguiló, M.A. Fiol, Double commutative-step digraphs with minimum diameters, *Discrete Math.* **114** (1993) 147–157.
- [4] D.F. Hsu and X.D. Jia, Extremal problems in the construction of distributed loop networks, *SIAM J. Discrete Math.* **7** (1994) 57–71.
- [5] A.G. Mask, J. Schneider, X. Jia, Extremal Cayley digraphs of finite Abelian groups, *J. Intercon. Networks* **12** (2011), no. 1-2, 125–135.
- [6] P. Morillo, M.A. Fiol, and J. Fàbrega, The diameter of directed graphs associated to plane tessellations, *Ars Combin.* **20-A** (1985) 17–27.
- [7] M.A. Fiol, J.L.A. Yebra, I. Alegre, M. Valero, A discrete optimization problem in local networks and data alignment, *IEEE Trans. Comput.* **C-36** (1987) 702–713.
- [8] C.K. Wong, D. Coppersmith, A combinatorial problem related to multinode memory organizations, *J. Assoc. Comput. Machin.* **21** (1974) 392–402.